

## Authors' Note

This is a corrected version of the article

M. E. H. Ismail and M. E. Muldoon, *Inequalities and monotonicity properties for gamma and  $q$ -gamma functions*, pp. 309-323 in R. V. M. Zahar, ed., *Approximation and Computation: A Festschrift in Honor of Walter Gautschi*, ISNM, vol. 119, Birkhäuser, Boston-Basel-Berlin, 1994.

Most of the errors in the original paper had to do with saying that certain functions related to the  $q$ -gamma function were **not** completely monotonic. We discovered these errors through reading the paper *Some completely monotonic functions involving the  $q$ -gamma function*, by Peng Gao, <http://arxiv.org/abs/1011.3303>.

We also take the opportunity to correct some errors in other places including the statement and proof of Theorem 3.4.

Corrected July 31, 2011

# INEQUALITIES AND MONOTONICITY PROPERTIES FOR GAMMA AND $q$ -GAMMA FUNCTIONS

Mourad E. H. Ismail<sup>1</sup> Martin E. Muldoon<sup>2</sup>

*To Walter Gautschi on his 65th birthday*

**Abstract** We prove some new results and unify the proofs of old ones involving complete monotonicity of expressions involving gamma and  $q$ -gamma functions,  $0 < q < 1$ . Each of these results implies the infinite divisibility of a related probability measure. In a few cases, we are able to get simple monotonicity without having complete monotonicity. All of the results lead to inequalities for these functions. Many of these were motivated by the bounds in a 1959 paper by Walter Gautschi. We show that some of the bounds can be extended to complex arguments.

## 1 INTRODUCTION AND PRELIMINARIES

Among Walter Gautschi's many contributions to mathematics are some interesting inequalities for the gamma function. For example, he shows ([10], [11]) that if  $x_k > 0, k = 1, \dots, n, x_1 x_2 \cdots x_n = 1$ , then the inequality

$$\sum_{k=1}^n \frac{1}{\Gamma(x_k)} \leq n \quad (1.1)$$

is true for  $n = 1, 2, \dots, 8$  but not for  $n \geq 9$ . Here we will be more concerned with an earlier result of Gautschi's [9], the two-sided inequality

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp[(1-s)\psi(n+1)], \quad 0 < s < 1, \quad n = 1, 2, \dots \quad (1.2)$$

which still inspires extensions. For example, D. Kershaw [13] proved

$$\exp[(1-s)\psi(x+s^{1/2})] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp[(1-s)\psi(x+(s+1)/2)], \quad 0 < s < 1, \quad x > 0, \quad (1.3)$$

and

$$\left[x + \frac{s}{2}\right]^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x - \frac{1}{2} + \left(s + \frac{1}{4}\right)^{\frac{1}{2}}\right]^{1-s}, \quad 0 < s < 1, \quad x > 0. \quad (1.4)$$

---

<sup>1</sup>Department of Mathematics, University of South Florida, Tampa, FL 33620-5700, U. S. A.

<sup>2</sup> Department of Mathematics and Statistics, York University, North York, Ont. M3J 1P3, Canada

In all of these inequalities,  $\psi(x)$  denotes the logarithmic derivative  $\Gamma'(x)/\Gamma(x)$ .

Many inequalities for special functions follow from monotonicity properties. Often such inequalities are special cases of the complete monotonicity of related special functions. For example, an inequality of the form  $f(x) \geq g(x)$ ,  $x \in [a, \infty)$  with equality if and only if  $x = a$ , may be a disguised form of the complete monotonicity of  $g(\varphi(x))/f(\varphi(x))$  where  $\varphi$  is a nondecreasing function on  $(a, \infty)$  and  $g(\varphi(a))/f(\varphi(a)) = 1$ . Thus, for example, the left-hand inequality in (1.2) and the right-hand one in (1.3) follow from the facts that  $x^s\Gamma(x+s)/\Gamma(x+1)$  and  $\exp[(s-1)\psi(x+(s+1)/2)]\Gamma(x+s)/\Gamma(x+1)$  are, respectively, decreasing and increasing functions of  $x$  on  $(0, \infty)$ . Bustoz and Ismail [4] proved that some of the above inequalities for the gamma function follow from the complete monotonicity of certain functions involving the ratio  $\Gamma(x+1)/\Gamma(x+s)$ .

Recall that a function  $f$  is completely monotonic on an interval  $I$  if

$$(-1)^n f^{(n)}(x) \geq 0$$

for  $n = 1, 2, \dots$  on  $I$ . We collect some known facts, all either easily proved or contained in [21] or [7], in the following theorem.

**Theorem 1.1.** (i) A necessary and sufficient condition that  $f(x)$  should be completely monotonic on  $(0, \infty)$  is that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where  $\alpha(t)$  is nondecreasing and the integral converges for  $0 < x < \infty$ .

(ii)  $e^{-h(x)}$  is completely monotonic on  $I$  if  $h'(x)$  is completely monotonic on  $I$ .

(iii) A probability distribution supported on a subset of  $[0, \infty)$  is infinitely divisible if and only if its Laplace transform (moment generating function) is of the form  $e^{-h(x)}$  with  $h(0^+) = 0$  and  $h'(s)$  is completely monotonic on  $(0, \infty)$ .

For brevity, we shall use *completely monotonic* to mean *completely monotonic* on  $(0, \infty)$ .

There is already an extensive and rich literature on inequalities for gamma functions; for references see [16], [17]. One of the objects of the present work is to show that many of these can be extended, using essentially the same methods of proof, to the  $q$ -gamma function defined (see, e.g., [8]) by

$$\Gamma_q(x) := (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}}, \quad 0 < q < 1. \quad (1.5)$$

Although the right-hand side of (1.5) is meaningful when  $|q| < 1$ , our results will require  $q^x \in (0, 1)$  for all positive  $x$ . This forces  $q \in (0, 1)$ . As expected

$$\Gamma_q(x) \rightarrow \Gamma(x) \quad \text{as } q \rightarrow 1^-. \quad (1.6)$$

The most elegant proof of this, due to R. William Gosper, is in Appendix A of Andrews's excellent monograph [3]; see also [8, p. 17]. For a rigorous justification, see [14]. It is worth noting that

$$\Gamma_q(x) \approx (1-q)^{1-x} \prod_0^{\infty} (1-q^{n+1}), \quad \text{as } x \rightarrow \infty \text{ if } |q| < 1. \quad (1.7)$$

It seems that most of our results have analogues also for the  $q$ -gamma function with  $q > 1$ , in which case the definition (1.5) must be changed. We do not pursue this question here. For the gamma function, we have the (Mittag-Leffler) sum representation [6]

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{z+n} \right), \quad (1.8)$$

and the integral representation [6, (1.7.14)]

$$\psi(x) = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-tx}}{1 - e^{-t}} dt, \quad \operatorname{Re} x > 0. \quad (1.9)$$

Although (1.9) and (1.8) are equivalent, for  $\operatorname{Re} x > 0$ , it turns out that (1.9) is more useful in proving the kind of inequalities in which we are interested. This situation occurred also in [20].

A corresponding sum representation for the case of the  $q$ -gamma function, easily following from (1.5) is

$$\psi_q(x) := \Gamma'_q(x)/\Gamma_q(x) = -\log(1-q) + \log q \sum_{n=0}^{\infty} q^{n+x}/(1-q^{n+x}), \quad 0 < q < 1. \quad (1.10)$$

Although this representation has been used directly in the proofs of many results for the  $q$ -gamma function (in [4], for example) we shall find it more convenient to use the equivalent Stieltjes integral representation

$$\psi_q(x) = -\log(1-q) - \int_0^{\infty} \frac{e^{-xt}}{1 - e^{-t}} d\gamma_q(t), \quad 0 < q < 1, \quad x > 0, \quad (1.11)$$

where  $d\gamma_q(t)$  is a discrete measure with positive masses  $-\log q$  at the positive points  $-k \log q$ ,  $k = 1, 2, \dots$ . For completeness, and economy of later statements, we include the value  $q = 1$  in the definition of  $\gamma_q(t)$ :

$$\gamma_q(t) = \begin{cases} -\log q \sum_{k=1}^{\infty} \delta(t + k \log q), & 0 < q < 1, \\ t, & q = 1. \end{cases} \quad (1.12)$$

To get the representation (1.11), we expand the denominator of the sum in (1.10) by the binomial theorem and interchange the orders of summation to get

$$\psi_q(x) = -\log(1-q) + \log q \sum_{k=1}^{\infty} q^{kx}/(1-q^k), \quad (1.13)$$

which is equivalent to (1.11).

Note that  $\psi_q(x)$  can also be expressed as a  $q$ -integral [8, p. 19 ],

$$\psi_q(x) = -\log(1-q) + \frac{\log q}{1-q} \int_0^1 \frac{t^{x-1}}{1-t} d_q(t), \quad 0 < q < 1, \quad x > 0, \quad (1.14)$$

just as, from (1.9),  $\psi(x)$  can be expressed as an ordinary integral over  $[0, 1]$ :

$$\psi(x) = -\gamma + \int_0^1 \frac{1-t^{x-1}}{1-t} dt, \quad \operatorname{Re} x > 0. \quad (1.15)$$

We will need the following relations which follow easily from the definition of  $d\gamma_q(t)$ :

$$\int_0^\infty e^{-xt} d\gamma_q(t) = \frac{-q^x \log q}{1-q^x}, \quad 0 < q < 1, \quad x > 0, \quad (1.16)$$

and

$$\int_0^\infty \frac{e^{-xt}}{t} d\gamma_q(t) = \sum_{k=1}^\infty \frac{q^{kx}}{k} = -\log(1-q^x), \quad 0 < q < 1, \quad x > 0. \quad (1.17)$$

We will use the following Lemma in many of our proofs. We remark that it includes results stated in different notations [4, Lemma 3.1] and [12, Lemma 4.1], as well as individual steps proved by *ad hoc* methods in these and other papers.

**Lemma 1.2.** *Let  $0 < \alpha < 1$ . Then*

$$\alpha e^{(\alpha-1)t} < \frac{\sinh \alpha t}{\sinh t} < \alpha, \quad t > 0. \quad (1.18)$$

*The inequalities become equalities when  $\alpha = 1$  and they are reversed when  $\alpha > 1$ .*

The following lemma will also be useful.

**Lemma 1.3.** (i) *Let  $f(x)$  be completely monotonic on  $(0, \infty)$  and let  $a > 0$ . Then  $f(x) - f(x+a)$  is completely monotonic on  $(0, \infty)$ . (ii) Let  $f(x) \geq 0$  and let  $f(x) - f(x+a)$  be completely monotonic on  $(0, \infty)$  for each  $a$  in some right-hand neighbourhood of 0. Then  $f(x)$  is completely monotonic on  $(0, \infty)$ .*

**Proof:** (i) We have

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where  $\alpha(t)$  is nondecreasing and the integral converges for  $0 < x < \infty$ . Hence

$$(-1)^n D_x^n [f(x) - f(x+a)] = \int_0^\infty [e^{-xt} - e^{-(x+a)t}] t^n d\alpha(t) \geq 0.$$

(ii) Under the hypotheses here, we find that  $-f'(x) = \lim_{a \rightarrow 0^+} [f(x) - f(x+a)]/a$  is completely monotonic on  $(0, \infty)$ . ■

**Remark 1.** A feature of the present work is that the similarity between (1.9) and (1.11) makes it possible to unify the proofs of some of our results for the gamma and  $q$ -gamma functions.

**Remark 2.** The integral representation in Theorem 1.1 (i) provides a necessary as well as a sufficient condition for the complete monotonicity of  $f$ . This enables us to show that certain monotonic functions are *not* completely monotonic more easily than is done in [2], for example. As in [12], many of our results will assert the complete monotonicity of a function for a certain range of values of a parameter, the complete monotonicity of its derivative for another range, and, **in some cases**, the complete monotonicity of neither of these for an intermediate range.

**Remark 3.** In many discussions of completely monotonic functions, the concept of *strict complete monotonicity* is used to indicate strict inequality in  $(-1)^n f^{(n)}(x) \geq 0$ . But if, as here, our interval is a half-line, we get such strict inequality in all but trivial cases: A result of J. Dubourdieu [5, p. 98] asserts that for a completely monotonic function on  $(a, \infty)$ , we have  $(-1)^n f^{(n)}(x) > 0$  for  $n = 1, 2, \dots$ , unless  $f(x)$  is constant.

**Remark 4.** In §5, we extend some bounds for ratios of gamma functions to complex values of the arguments.

## 2 GAMMA AND $q$ -GAMMA FUNCTIONS

The following result was proved in [12]:

**Theorem 2.1.** *Let  $h_\alpha(x) = \log[x^\alpha \Gamma(x)(e/x)^x]$ . Then  $-h'_\alpha(x)$  is completely monotonic on  $(0, \infty)$  for  $\alpha \leq 1/2$ ,  $h'_\alpha(x)$  is completely monotonic for  $\alpha \geq 1$ , and neither is completely monotonic for  $1/2 < \alpha < 1$ .*

The proof in [12] is based on the consequence

$$-h'_\alpha(x) = \int_0^\infty \left[ \frac{1}{1-e^{-t}} - \frac{1}{t} - \alpha \right] e^{-xt} dt,$$

of (1.9) and the fact that the quantity in the square brackets, which has the same sign as  $(1+\alpha t)e^{-t} - 1 + (1-\alpha t)$ , is positive for  $\alpha \leq 1/2$ , negative for  $\alpha \geq 1$  and undergoes a change of sign for  $1/2 < \alpha < 1$ . The next result can be considered a  $q$ -analogue of Theorem 2.1.

**Theorem 2.2.** *Let  $0 < q < 1$  and let*

$$h_\alpha(x) = \log \left[ (1-q)^x (1-q^x)^\alpha \Gamma_q(x) \exp \left( \sum_{k=1}^{\infty} q^{kx} / (k^2 \log q) \right) \right].$$

*Then  $-h'_\alpha(x)$  is completely monotonic on  $(0, \infty)$  for  $\alpha \leq 1/2$  and  $h'_\alpha(x)$  is completely monotonic on  $(0, \infty)$  for  $\alpha \geq 1$ .*

**Proof:** It follows from (1.11) that

$$-h'_\alpha(x) = \int_0^\infty \left[ \frac{1}{1-e^{-t}} - \frac{1}{t} - \alpha \right] e^{-xt} d\gamma_q(t),$$

where  $d\gamma_q(t)$  is defined by (1.12). As in the proof of [12, Theorem 2.1], the quantity in square brackets is positive for  $\alpha \leq \frac{1}{2}$  and negative for  $\alpha \geq 1$ . ■

**Remark.** To see that Theorem 2.1 includes the limiting case of Theorem 2.2 as  $q \rightarrow 1^-$ , we will show that

$$\lim_{q \rightarrow 1^-} \frac{(1-q)^x (1-q^x)^\alpha \Gamma_q(x) \exp[F(q^x)/\log q]}{(1-q)^\alpha \exp[F(1)/\log q]} = x^\alpha \Gamma(x) (e/x)^x,$$

where

$$F(x) = \sum_{n=1}^\infty \frac{x^n}{n^2} = - \int_0^x \frac{\log(1-t)}{t} dt. \quad (2.1)$$

There is no difficulty in seeing that

$$\lim_{q \rightarrow 1^-} \frac{(1-q^x)^\alpha \Gamma_q(x)}{(1-q)^\alpha} = x^\alpha \Gamma(x),$$

so it remains to show that

$$\lim_{q \rightarrow 1^-} \frac{(1-q)^x \exp[F(q^x)/\log q]}{\exp[F(1)/\log q]} = (e/x)^x.$$

Taking logarithms, this is equivalent to showing that

$$\lim_{q \rightarrow 1^-} \left[ x \log(1-q) + \frac{F(q^x) - F(1)}{\log q} \right] = x - x \log x,$$

and this follows from the identity

$$x \log(1-q) + \frac{F(q^x) - F(1)}{\log q} \equiv - \int_0^x \log \frac{1-q^t}{1-q} dt,$$

which is easily checked by differentiation with respect to  $x$ .

Applying Lemma 1.3 to the results of the last theorem, we get:

**Theorem 2.3.** *Let  $0 < q < 1$ ,  $a > 0$  and let*

$$H_\alpha(x) = \log \left[ \left( \frac{1-q^x}{1-q^{x+a}} \right)^\alpha \frac{\Gamma_q(x)}{\Gamma_q(x+a)} \exp \left( \sum_{k=1}^\infty \frac{q^{kx} - q^{k(x+a)}}{k^2 \log q} \right) \right].$$

*Then  $-H'_\alpha(x)$  is completely monotonic on  $(0, \infty)$  for  $\alpha \leq 1/2$  and  $H'_\alpha(x)$  is completely monotonic on  $(0, \infty)$  for  $\alpha \geq 1$ .*

In the case  $q \rightarrow 1^-$ , the following Corollary follows by applying Lemma 1.3 to the result of Theorem 2.1.

**Corollary 2.4.** *Let  $a > 0$  and let*

$$H_\alpha(x) = \log \left( \frac{x^{\alpha-x} \Gamma(x)}{(x+a)^{\alpha-x-a} \Gamma(x+a)} \right).$$

*Then  $-H_\alpha'(x)$  is completely monotonic on  $(0, \infty)$  for  $\alpha \leq 1/2$ ,  $H_\alpha'(x)$  is completely monotonic on  $(0, \infty)$  for  $\alpha \geq 1$  and neither is completely monotonic, for  $1/2 < \alpha < 1$ .*

The Corollary introduces ratios of gamma functions whose asymptotic behavior [1, 6.1.47]

$$z^{b-a} \frac{\Gamma(z+a)}{\Gamma(z+b)} \sim 1 + \frac{(a-b)(a+b+1)}{2z} + \dots \quad (2.2)$$

suggests that we look at the possible completely monotonic character of such ratios for both gamma and  $q$ -gamma functions.

Our first result is suggested by [4, Theorem 3] on  $(x+c)^{a-b} \Gamma(x+b)/\Gamma(x+a)$ :

**Theorem 2.5.** *Let  $a < b \leq a+1$  and let*

$$g(x) := \left[ \frac{1-q^{x+c}}{1-q} \right]^{a-b} \frac{\Gamma_q(x+b)}{\Gamma_q(x+a)}. \quad (2.3)$$

*Then  $-(\log g(x))'$  is completely monotonic on  $(-c, \infty)$  if  $0 \leq c \leq (a+b-1)/2$  and  $(\log g(x))'$  is completely monotonic on  $(-a, \infty)$  if  $c \geq a \geq 0$ . Remarks.* The limiting case  $q \rightarrow 1^-$  of this theorem was proved by Bustoz and Ismail [4, Theorem 3]. (Note that in the statement of [4, Theorem 3 (ii)] “ $x > \beta$ ” should read “ $x > \alpha$ ”.) Although, by a translation of the variable  $x$ , one could assume  $a = 0$  and so express the theorem in a simpler form, we prefer to retain the  $a$  and  $b$  for reasons of symmetry. An interesting consequence of the last assertion of the theorem is that neither  $h'$  or  $-h'$  is completely monotonic when

$$c = -\frac{1}{2} + \sqrt{ab + \frac{1}{4}},$$

and hence  $(a+b-1)/2 < c < a$ . Thus although it is true that

$$\frac{d}{dx} \log \left[ \frac{\Gamma(x+a)}{\Gamma(x+b)} \left( x - \frac{1}{2} + \sqrt{ab + \frac{1}{4}} \right)^{b-1} \right] \quad (2.4)$$

is positive (a result motivated by the right-hand inequality in (1.4), and proved essentially by the method given for the case  $b = 1$  in [4, Theorem 8]), it is not completely monotonic.



*Proof of Theorem 2.5.* We note that

$$\begin{aligned}\frac{d}{dx} \log g(x) &= (b-a) \frac{q^{x+c} \log q}{1-q^{x+c}} + \psi_q(x+b) - \psi_q(x+a) \\ &= - \int_0^\infty e^{-xt} \left[ \frac{e^{-bt} - e^{-at}}{1-e^{-t}} + (b-a)e^{-ct} \right] d\gamma_q(t),\end{aligned}$$

where  $\gamma_q$  is given by (1.12). Now we will use Lemma 1.2, to show that the quantity in square brackets has the appropriate sign. In case  $c \leq (a+b-1)/2$ , we find that the integrand is  $e^{-(x+c)t}$  times a function of  $t$  which exceeds

$$b-a - \frac{\sinh[(b-a)t/2]}{\sinh(t/2)}$$

and this is positive by Lemma 1.2. When  $c \geq a$ , the integrand is  $e^{-(x+a)t}$  times a function of  $t$  which is less than

$$(b-a)e^{(b-a-1)t/2} - \frac{\sinh[(b-a)t/2]}{\sinh(t/2)}$$

and this is negative by Lemma 1.2. On the other hand, if  $(a+b-1)/2 < c < a$ , the quantity in square brackets is positive for  $t$  close to 0 and negative for large  $t$ . ■

The ranges of  $a, b$  and  $c$  in Theorem 2.5 fail to cover some interesting cases. For example, when  $b = c = 0$ , it gives  $-\log g(x)$  completely monotonic for  $a \leq 0$ , but gives no information for  $a > 0$ . Hence it is of interest to record the following result:

**Theorem 2.6.** *Let  $0 < q < 1$  and let*

$$h(x) = \log \left[ \left( \frac{1-q^x}{1-q} \right)^a \frac{\Gamma_q(x)}{\Gamma_q(x+a)} \right]. \quad (2.5)$$

*Then  $h'(x)$  is completely monotonic on  $(0, \infty)$  for  $a \geq 1$ .*

**Proof:** From (1.11), we have

$$\begin{aligned}h'(x) &= \int_0^\infty e^{-xt} \left[ \frac{e^{-at}-1}{1-e^{-t}} + a \right] d\gamma_q(t) \\ &= \int_0^\infty e^{-xt+t(1-a)/2} \left[ ae^{(a-1)t/2} - \frac{\sinh(at/2)}{\sinh(t/2)} \right] d\gamma_q(t)\end{aligned} \quad (2.6)$$

and we see from Lemma 2.2 that the quantity in square brackets is positive for  $a > 1$ . ■

### 3 PSI AND $q$ -PSI FUNCTIONS

The psi function has particularly simple monotonicity properties. For example, it follows from (1.8) that

$$\psi'(x) = \sum_{n=0}^{\infty} (n+x)^{-2}, \quad (3.1)$$

so  $\psi'(x)$  is completely monotonic on  $(0, \infty)$ . G. Ronning [20] showed that, for  $0 < \alpha < 1$ ,  $\psi'(x) - \alpha\psi'(\alpha x) < 0$ ,  $0 < x < \infty$ . More generally, we show:

**Theorem 3.1.** *Let  $0 < \alpha < 1$ ,  $0 < q < 1$ . Then the function  $\psi_q(x) - \psi_{q^{1/\alpha}}(\alpha x)$  is completely monotonic, i.e.,*

$$(-1)^n [\psi_q^{(n)}(x) - \alpha^n \psi_{q^{1/\alpha}}^{(n)}(\alpha x)] > 0, \quad 0 < x < \infty, \quad n = 0, 1, \dots \quad (3.2)$$

**Proof:** Using (1.9) and (1.11) we see that

$$\psi_q(x) - \psi_{q^{1/\alpha}}(\alpha x) = \frac{1}{\alpha} \int_0^{\infty} e^{-xt} \left[ -\frac{\alpha}{1-e^{-t}} + \frac{1}{1-e^{-t/\alpha}} \right] d\gamma_q(t). \quad (3.3)$$

The quantity in square brackets is seen to be equal to

$$\frac{\alpha e^{(1-1/\alpha)t/2}}{1-e^{-t/\alpha}} \left[ \frac{1}{\alpha} e^{(t/2)(1/\alpha-1)} - \frac{\sinh(t/(2\alpha))}{\sinh(t/2)} \right],$$

and the term in square brackets is seen to be positive on using the left-hand inequality in Lemma 1.2 (with  $\alpha$  replaced by  $1/\alpha$ ). The result follows.  $\blacksquare$

Gautschi's and Kershaw's inequalities (1.2) and (1.3) suggest that we consider ratios of the form

$$\frac{\Gamma_q(x+a)}{\Gamma_q(x+b)} \exp[(b-a)\psi_q(x+c)]. \quad (3.4)$$

We have the following result.

**Theorem 3.2.** *Let  $0 < a < b$ ,  $0 < q \leq 1$  and let  $h(x)$  denote the logarithm of the function in (3.4). Then, if  $c \geq (a+b)/2$ ,  $-h'(x)$  is completely monotonic on  $(-a, \infty)$  and if  $c \leq a$ ,  $h'(x)$  is completely monotonic on  $(-c, \infty)$ . Neither  $h'$  or  $-h'$  is completely monotonic for  $a < c < (a+b)/2$ .*

Bustoz and Ismail have this result in the case  $q = 1$ ,  $b = 1$ ,  $c = (a+b)/2$ .

**Proof:** We have, from (1.9) and (1.11),

$$h'(x) = - \int_0^{\infty} \frac{e^{-(x+(a+b)/2)t}}{1-e^{-t}} \left[ 2 \sinh \frac{b-a}{2} t - (b-a) t e^{((a+b)/2-c)t} \right] d\gamma_q(t). \quad (3.5)$$

Using the inequality  $\sinh \theta > \theta$ ,  $\theta > 0$ , we see that the quantity in square brackets is positive when  $c \geq (a+b)/2$ . When  $c \leq a$ , the quantity in square brackets is

negative on account of the inequality  $e^{-2x} > 1 - 2x$ ,  $x > 0$ . On the other hand, when  $a < c < (a+b)/2$  this quantity is negative for small  $t$  and positive for large  $t$ . This completes the proof. An interesting consequence of the last assertion of the theorem is that neither  $h'$  or  $-h'$  is completely monotonic when  $c$  is the geometric mean of  $a$  and  $b$ . Thus although it is true that

$$\frac{d}{dx} \log \left[ \frac{\Gamma(x+a)}{\Gamma(x+b)} \exp \left[ (b-a)\psi(x+\sqrt{ab}) \right] \right] \quad (3.6)$$

is positive (the result is suggested by the left-hand inequality in (1.3) and the proof is essentially that given for the case  $b = 1$  in [4, Theorem 8]), neither it nor its negative is completely monotonic. ■

A  $q$ -analogue of inequalities in (1.3) and (1.4) runs as follows:

**Theorem 3.3.** *For  $0 < q \leq 1$ , we have*

$$\left( \frac{1 - q^{x+s/2}}{1 - q} \right)^{1-s} < \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} < \exp[(1-s)\psi_q(x+(s+1)/2)], \quad 0 < s < 1, \quad (3.7)$$

where the left-hand inequality holds for  $x > -s/2$  and the right-hand one holds for  $x > -s$ .

**Proof:** Theorem 2.5 shows that

$$\left[ \frac{1 - q^{x+s/2}}{1 - q} \right]^{s-1} \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)}$$

decreases on  $(-s/2, \infty)$  to its limiting value 1. This gives the left-hand part of (3.7). Theorem 3.3 shows that

$$\exp(1-s)\psi_q \left( x + \frac{s+1}{2} \right) \frac{\Gamma_q(x+s)}{\Gamma_q(x+1)}$$

decreases on  $(-s, \infty)$  to its limiting value 1. This gives the right-hand part of (3.7).

A result of H. Alzer [2, Theorem 1] suggests dealing with products of  $q$ -gamma functions and exponentials of derivatives of  $q$ -psi functions. In this connection, we prove:

**Theorem 3.4** *Let  $0 < q < 1$ ,  $0 < s < 1$  and*

$$g_\alpha(x) = (1-q)^x (q^{-x} - 1) \Gamma_q(x) \exp \left[ F(q^x) / \log q - \frac{1}{12} \psi'_q(x+\alpha) \right]$$

where  $F$  is given by (2.1). Then  $(\log g_\alpha)'$  is completely monotonic on  $(0, \infty)$  for  $\alpha \geq 1/2$ ,  $-(\log g_\alpha)'$  is completely monotonic on  $(0, \infty)$  for  $\alpha \leq 0$ .

*Proof.* We have

$$\frac{d}{dx} \log g_\alpha(x) = - \int_0^\infty e^{-xt} p_\alpha(t) d\gamma_q t$$

where

$$p_\alpha(t) = \frac{12 - t^2 e^{-\alpha t}}{12(1 - e^{-t})} - \frac{1}{2} - \frac{1}{t}$$

Now, for  $\alpha \geq 1/2$ , we have  $p_\alpha(t) > 0$  for  $t > 0$  [2, p. 339]. Also when  $\alpha \leq 0$ , we have  $p_\alpha(t) < 0$  for  $t > 0$ . Thus we get the required complete monotonicity properties of  $g_\alpha$ .

If we combine this Theorem with Lemma 1.3 we get the following extension of [2, Theorem 1] to which it reduces when  $q \rightarrow 1^-$ .

**Corollary 3.5.** *Let  $0 < q < 1$ ,  $0 < s < 1$  and*

$$\begin{aligned} f_\alpha(x) &= g_\alpha(x+s)/g_\alpha(x+1) \\ &= \left[ \frac{(1-q)^{s-1}(1-q^{x+s})^{1/2}\Gamma_q(x+s)}{(1-q^{x+1})^{1/2}\Gamma_q(x+1)} \right. \\ &\quad \left. \times \exp \left\{ \frac{F(q^{x+s}) - F(q^{x+1})}{\log q} + \frac{1}{12} [\psi'_q(x+1+\alpha) - \psi'_q(x+s+\alpha)] \right\} \right]. \end{aligned}$$

*Then  $(\log f)'$  is completely monotonic on  $(0, \infty)$  for  $\alpha \geq 1/2$ ,  $-(\log f)'$  is completely monotonic on  $(0, \infty)$  for  $\alpha \leq 0$ , and neither is completely monotonic on  $(0, \infty)$  for  $0 < \alpha < 1/2$ .*

The limiting case  $q \rightarrow 1^-$  of this last Corollary runs as follows:

**Corollary 3.6.** *Let  $0 < s < 1$  and*

$$\begin{aligned} f_\alpha(x) &= \frac{(x+1)^{x+1/2}\Gamma_q(x+s)}{(x+s)^{x+s-1/2}\Gamma_q(x+1)} \\ &\quad \times \exp \left\{ s - 1 + \frac{1}{12} [\psi'(x+1+\alpha) - \psi'(x+s+\alpha)] \right\}. \end{aligned}$$

*Then  $(\log f)'$  is completely monotonic on  $(0, \infty)$  for  $\alpha \geq 1/2$ ,  $-(\log f)'$  is completely monotonic on  $(0, \infty)$  for  $\alpha \leq 0$ , and neither is completely monotonic on  $(0, \infty)$  for  $0 < \alpha < 1/2$ .*

This recovers, in a slightly extended form, the main result [2, Theorem 1] of H. Alzer.

## 4 FURTHER PRODUCTS AND QUOTIENTS

Results of Bustoz and Ismail [4, Theorem 6] concerning the ratio

$$\frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x)\Gamma(x+a+b)},$$

suggest the consideration of ratios

$$\frac{\Gamma_q(x+a_1)\Gamma_q(x+a_2)\dots\Gamma_q(x+a_n)}{\Gamma_q(x+b_1)\Gamma_q(x+b_2)\dots\Gamma_q(x+b_n)}, \quad (4.1)$$

where

$$a_1 + a_2 + \dots + a_n = b_1 + b_2 + \dots + b_n = s. \quad (4.2)$$

If we treat the  $a_i$  as given, each choice of the  $b_i$  may be thought of as a partition of the sum of the  $a_i$ . An extreme case would consist of taking the  $b_i$  equal to each other; another extreme case would be to take all except one of the  $b_i$  to be 0. Both of these lead to completely monotonic functions:

**Theorem 4.1.** *Let  $a_1, \dots, a_n$  be positive numbers, let  $n\bar{a} = a_1 + \dots + a_n$ , and  $0 < q \leq 1$ . Then both*

$$-\frac{d}{dx} \log \frac{\Gamma_q(x+a_1)\Gamma_q(x+a_2)\dots\Gamma_q(x+a_n)}{\Gamma_q(x+\bar{a})^n} \quad (4.3)$$

and

$$\frac{d}{dx} \log \frac{\Gamma_q(x+a_1)\Gamma_q(x+a_2)\dots\Gamma_q(x+a_n)}{\Gamma_q(x)^{n-1}\Gamma_q(x+a_1+a_2+\dots+a_n)} \quad (4.4)$$

are completely monotonic on  $(0, \infty)$ .

**Proof:** Using (1.9) and (1.11), we find that these expressions may be written

$$\int_0^\infty \frac{e^{-xt}}{1-e^{-t}} [ne^{-\bar{a}t} - e^{-a_1t} - \dots - e^{-a_nt}] d\gamma_q(t) \quad (4.5)$$

and

$$\int_0^\infty \frac{e^{-xt}}{1-e^{-t}} \left[ n-1 + e^{-(a_1+\dots+a_n)t} - e^{-a_1t} - \dots - e^{-a_nt} \right] d\gamma_q(t) \quad (4.6)$$

In the first case the quantity in square brackets is positive by Jensen's theorem while in the second case its positivity follows from

$$n-1 + z_1z_2\dots z_n - (z_1 + z_2 + \dots + z_n) \geq 0, \quad 0 \leq z_i < 1. \quad (4.7)$$

This is clear since it may be established by induction on  $n$  that

$$n-1 + z_1z_2\dots z_n - (z_1 + z_2 + \dots + z_n) = \sum_{j=2}^n (1-z_j)(1-z_1z_2\dots z_{j-1}). \quad (4.8)$$

■

The positivity of the quantity (34) leads to the inequality

$$\Gamma_q(1 + a_1)\Gamma_q(1 + a_2)\dots\Gamma_q(1 + a_n) \geq \Gamma_q(1 + \bar{a})^n, \quad a_i > 0,$$

which is known already in the case  $q = 1$  [17, p. 285].

## 5 BOUNDS IN THE COMPLEX PLANE

A generalization of the Phragmén–Lindelöf theorem is used in [19, pp. 68–70] to show that

$$\left| \frac{\Gamma(s + c)}{\Gamma(s)} \right| \leq |s|^c, \quad 0 \leq c \leq 1, \quad \operatorname{Re}(s) \geq (1 - c)/2. \quad (5.1)$$

Here we show that the same method can be used to get bounds for more complicated functions involving gamma functions.

The Phragmén–Lindelöf theorem runs as follows [19, p. 59]:

**Theorem 5.1.** *Let  $f(z)$  be analytic in the strip  $S(\alpha, \beta) = \{z | z = x + iy, \alpha < x < \beta\}$ . Let us assume  $|f(z)| \leq 1$  on the boundaries  $x = \alpha$  and  $x = \beta$  and moreover,*

$$|f(z)| < Ce^{k|y|}$$

*for some  $C > 0$  and  $0 < k < \pi/(\beta - \alpha)$ . Then  $|f(z)| \leq 1$  throughout the strip  $S(\alpha, \beta)$ .*

By Theorem 4.1, the function

$$f(x) := \frac{\Gamma(x + a)\Gamma(x + b)}{\Gamma(x)\Gamma(x + a + b)}, \quad a, b \geq 0. \quad (5.2)$$

is increasing on the interval  $(0, \infty)$  to its limit 1. Hence we have

$$\left| \frac{\Gamma(x + a)\Gamma(x + b)}{\Gamma(x)\Gamma(x + a + b)} \right| \leq 1, \quad x > 0. \quad (5.3)$$

We have, in fact:

**Theorem 5.2.** *We have, for  $0 \leq a \leq 1, b \geq 0$ ,*

$$\left| \frac{\Gamma(s + a)\Gamma(s + b)}{\Gamma(s)\Gamma(s + a + b)} \right| \leq 1, \quad \operatorname{Re} s > \frac{1 - a - b}{2}. \quad (5.4)$$

*Proof.* The method follows the proof of [19, Theorem A, p. 68]. Since the assertion is trivial for  $a = 0$ , we may as well choose  $a > 0$ . We choose a complex number  $s = \sigma + i\tau$  satisfying the hypotheses of the theorem and let

$$f(z) = \frac{\Gamma(a + 2\sigma - z)\Gamma(b + z)}{\Gamma(z)\Gamma(a + b + 2\sigma - z)}.$$

Clearly

$$|f(s)| = \left| \frac{\Gamma(s+a)\Gamma(s+b)}{\Gamma(s)\Gamma(s+a+b)} \right|$$

so we have to show that  $|f(s)| \leq 1$ . Now with  $\alpha = (a-1)/2 + \sigma$ ,  $\beta = a/2 + \sigma$ , and using  $\overline{\Gamma(z)} = \Gamma(\bar{z})$ , we get

$$|f(\alpha + it)| = \frac{|\alpha + it|}{|b + \alpha + it|} \leq 1$$

where we have used  $\operatorname{Re} s \geq (1-a-b)/2$ . Also

$$|f(\beta + it)| = 1,$$

and (20) shows that the growth condition of Theorem 5.1 is satisfied. Using this theorem, we find that  $|f(z)| \leq 1$ , for  $\alpha \leq \operatorname{Re} z \leq \beta$ , so, in particular,  $|f(s)| \leq 1$ .

**Acknowledgments.** The authors' work was supported partially by NSF grant DMS 9203659 and by NSERC Canada grant A5199. We are grateful to Ruiming Zhang for pointing out the relevance of reference [19]. We thank a referee for corrections and suggestions.

## References

- [1] M. Abramowitz and I. A. Stegun, eds. *Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables*. National Bureau of Standards, Applied Mathematics Series **55**, Washington, 1964.
- [2] H. Alzer. Some gamma function inequalities. *Math. Comp.* **60** (1993), 337–346.
- [3] G. E. Andrews. *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*. Regional Conference Series in Mathematics Number 66, American Mathematical Society, Providence, 1985.
- [4] J. Bustoz and M. E. H. Ismail. On gamma function inequalities. *Math. Comp.* **47** (1986), 659–667.
- [5] J. Dubourdieu. Sur un théorème de M. S. Bernstein relatif à la transformation de Laplace–Stieltjes. *Compositio Math.* **7** (1939–40), 96–111.
- [6] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi. *Higher Transcendental Functions*, vol. 1. McGraw-Hill, New York, 1953.
- [7] W. Feller. *An Introduction to Probability Theory and its Applications*, vol. 2. Wiley, New York, 1966.

- [8] G. Gasper and M. Rahman. *Basic Hypergeometric Series*. Cambridge University Press, Cambridge, 1990.
- [9] W. Gautschi. Some elementary inequalities relating to the gamma and incomplete gamma function. *J. Math. Phys.* **38** (1959), 77–81.
- [10] W. Gautschi. A harmonic mean inequality for the gamma function. *SIAM J. Math. Anal.* **5** (1974), 278–281.
- [11] W. Gautschi. Some mean value inequalities for the gamma function. *SIAM J. Math. Anal.* **5** (1974), 282–292.
- [12] M. E. H. Ismail, L. Lorch and M. E. Muldoon. Completely monotonic functions associated with the gamma function and its  $q$ -analogues. *J. Math. Anal. Appl.* **116** (1986), 1–9.
- [13] D. Kershaw. Some extensions of W. Gautschi’s inequalities for the gamma function. *Math. Comp.* **41** (1983), 607–611.
- [14] T. H. Koornwinder. Jacobi functions as limit cases of  $q$ -ultraspherical polynomials. *J. Math. Anal. Appl.* **148** (1990), 44–54.
- [15] A. Laforgia. Further inequalities for the gamma function. *Math. Comp.* **42** (1984), 597–600.
- [16] A. W. Marshal and I. Olkin. *Inequalities: Theory of Majorization and Applications*. Academic Press, New York, 1979.
- [17] D. S. Mitronović. *Analytic Inequalities*. Springer-Verlag, Berlin, 1971.
- [18] M. E. Muldoon. Some monotonicity properties and characterizations of the gamma function. *Aequationes Math.* **18** (1978), 54–63.
- [19] H. Rademacher. *Topics in Analytic Number Theory*. Springer-Verlag, Berlin–New York, 1973.
- [20] G. Ronning. On the curvature of the trigamma function. *J. Comp. Appl. Math.*, **15** (1986), 397–399.
- [21] D. V. Widder. *The Laplace Transform*. Princeton University Press, Princeton, 1941.